Problem Set 2 Solutions

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Problem 1: Accretion onto a supermassive black hole

Assume the BH has a mass $M_{BH} \sim 10^7\ M_{\odot}$, and radiates at its electron-scattering Eddington luminosity

$$L_{BH} = L_{Edd,es} = \frac{4\pi G M_{BH} c}{\kappa_{es}},$$

where $\kappa_{es} \approx 0.4\mathrm{cm}^2/\mathrm{g}$.

We want to model the source of BH radiation as an isotropically emitting sphere, neglecting limb darkening (ie a Lambert sphere), which has $R_{in} \sim 10R_S$ (where R_S is the Schwarzschild radius, $R_S = 2GM/c^2$).

The accreting BH is enshrouded by a spherical envelope of H with dust mixed in, extending out to a radius $R_{out} \sim 10\mathrm{pc}$ (so $R_{out} \gg R_{in}$).

We will assume that radiative heating/cooling dominates the energy exchange, so given a long enough time, the envelope will come into radiative equilibrium.

The optical depth of the envelope is $\tau_0 \approx \rho_0 \kappa R_{out}$, where $\kappa \sim 10\mathrm{cm}^2/\mathrm{g}$ is the IR opacity of dusty gas.

Optically thin limit: $\tau_0 \ll 1$

(a) Assume that the specific intensity from the surface of the internal source is given by the Planck function at some temperature T_s.

So we write $I_0(I \text{ at } R_{in}) = B(T_s)$, where

$$B(T_s) = \frac{\sigma_{SB} T_s^4}{\pi}$$

For an isotropically emitting surface at R_{in}, the luminosity comes from the total flux directed outwards from $\mu = 0 \rightarrow \mu = 1$ at R_{in}. So

$$F_{out} = \int_0^{2\pi} \int_0^1 I\,\mu\, d\mu\, d\phi = 2\pi I_0 \left.\frac{\mu^2}{2}\right|_0^1 = \pi I_0 = \sigma_{SB} T_s^4$$

So the total luminosity eminating from this surface is

$$L_{BH} = L_{Edd,es} = 4\pi R_{in}^2 \sigma_{SB} T_s^4$$

But

$$L_{BH} = 4\pi R_{in}^2 \sigma_{SB} T_s^4 = \frac{4\pi G M_{BH} c}{\kappa_{es}}$$

and $R_{in} = 10R_S = 20GM_{BH}/c^2 = 1.98\mathrm{AU} = 2.96 \times 10^{13}\mathrm{cm}$, so using $R_{in} = 20GM_{BH}/c^2$ we solve for T_s:

$$T_s = \left(\frac{c^5}{400\sigma_{SB} G M_{BH} \kappa_{es}}\right)^{1/4}$$

$$\boxed{T_s = 2.1 \times 10^5 \mathrm{K}}$$

Using Wien's law $\lambda_{pk} T = 0.3$ cm K, we find the accreting black hole radiates most at about

$$\boxed{\lambda_{pk} \approx 142 \text{ Å}} \qquad \text{(EUV)}$$

(b) Now we want to write down the mean intensity $J(r)$ for all $r \geq R_{in}$ and use this to solve for the temperature profile $T(r)$, assuming the envelope is in radiative equilibrium.

$$J = \frac{1}{4\pi} \oint I_0 \, d\mu \, d\phi$$

In this case, I is only non-zero for solid angles looking at the BH radiative surface.

As in lecture on 19 Jan, for the Lambert sphere case (ie our isotropically emitting sphere), we have

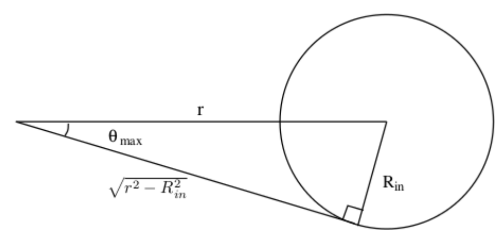
$$I = \begin{cases} I_0 & \theta < \theta_{max} \\ 0 & \theta > \theta_{max} \end{cases}$$

So

$$\begin{aligned} J &= \frac{1}{4\pi} \int_0^{\theta_{max}} \int_0^{2\pi} I_0 \, d\phi \, \sin\theta \, d\theta \\ &= \frac{2\pi}{4\pi} I_0 (-\cos\theta) \Big|_0^{\theta_{max}} \\ &= \frac{1}{2} I_0 \left(1 - \cos\theta_{max}\right) \end{aligned}$$

But

$$\begin{aligned} \cos\theta_{max} &= \frac{\sqrt{r^2 - R_{in}^2}}{r} \\ &= \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2} \end{aligned}$$



So

$$J(r) = \frac{1}{2} I_0 \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)$$

where $I_0 = B(T_s) = \sigma_{SB} T_s^4 / \pi$, so we get

$$\boxed{J(r) = \frac{\sigma_{SB} T_s^4}{2\pi} \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)}$$

Now if our dusty envelope is in radiative equilibrium, then $\dot{H}_\gamma = \dot{C}_\gamma$. Here, we're assuming the extinction is grey (and purely absorptive). Grey α implies that $T_{equil} = \left[\frac{\pi}{\sigma_{SB}} J\right]^{1/4}$

So $T(r) = \left[\frac{\pi}{\sigma_{SB}} J(r)\right]^{1/4}$ $\Rightarrow$

$$T(r) = \left[\frac{\cancel{\pi}}{\cancel{\sigma_{SB}}} \frac{\cancel{\sigma_{SB}} T_s^4}{2\cancel{\pi}} \left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)\right]^{1/4}$$

$$\boxed{T(r) = T_s \left[\frac{1}{2}\left(1 - \left[1 - \frac{R_{in}^2}{r^2}\right]^{1/2}\right)\right]^{1/4}}$$

Characteristic temperature for the envelope:

$$\boxed{T(r \sim R_{out}/2) = 210 \text{ K}},$$

and using Wien's law,

$$\boxed{\lambda_{pk} = 14\ \mu\text{m}} \qquad \text{(IR)}$$

(c) Check if the stationary approximation is okay.

For our optically thin dusty envelope,

$$t_{esc} \approx \frac{R}{\lambda_{mfp}} \frac{\lambda_{mfp}}{c} \approx \frac{R_{out}}{c}, \text{ since } R_{out} - R_{in} \approx R_{out} \text{ (ie } t_{esc} \approx \text{light-crossing time)}$$

$$\text{So} \quad \boxed{t_{esc} \approx \frac{R_{out}}{c} = 10^9 \text{ s} \approx 31.7 \text{ yr}}$$

Compare t_{esc} to the dynamical timescale: $t_{dyn} = t_{ff}$. The free fall time (assuming $M_{envelope} \ll M_{BH}$) is found using

$$\ddot{r} = \frac{-GM_{BH}}{r^2}$$

But we know that $\ddot{r} \approx r/{t_{ff}}^2$ (to OOM), so

$$\frac{r}{{t_{ff}}^2} \approx \left|\frac{GM_{BH}}{r^2}\right|$$

$$t_{ff} \approx \sqrt{\frac{{R_{out}}^3}{GM_{BH}}}$$

$$\text{So} \quad \boxed{t_{dyn} = t_{ff} \approx 4.5 \times 10^{12} \text{ s} \approx 1.4 \times 10^5 \text{ yr}}$$

So we find that $t_{dyn} \gg t_{esc}$.

Now consider t_{eq}, the time for the envelope to come into radiative equilibrium. As an estimate, use $t_{eq} \approx t_{cooling}$ or $t_{heating}$ at $\sim R_{out}/2$.

For the thermal cooling time, if we assume that the cooling process is through electron scattering, then $\sigma_{cooling} = \sigma_T$, and if we assume that we have an isotropic, grey, ideal gas envelope, then

$$t_{cooling} \approx \frac{3}{8} \frac{k}{\sigma \sigma_{SB} T^3}$$

$$\text{So} \quad t_{cooling}(R_{out}/2) \approx \frac{3}{8} \frac{k}{\sigma_T \sigma_{SB} T(R_{out}/2)^3}$$

$$\boxed{t_{cooling}(R_{out}/2) \approx 1.47 \times 10^5 \text{ s} \approx 5 \times 10^{-3} \text{ yr}}$$

and we find that $t_{cooling} \lll t_{dyn}$. Thus it does seem safe to assume the envelope structure is fixed over the timescale on which photons are escaping: thermal perturbations are very quickly washed out compared to any collapse, and the photons escape long before the envelope can appreciably collapse.

Optically thick limit: $\tau_0 \gg 1$

For the optically thick case, we need to use the diffusion approximation. We will still assume the envelope opacity is grey and purely absorptive.

The diffusion equation in spherical coordinates is

$$L(r) = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial}{\partial r} u(r)$$

Assuming we have radiative equilibrium, we know that $\frac{\partial}{\partial r}(r^2 F) \equiv \text{const}$, or rather that $L = 4\pi r^2 F \equiv \text{const}$ with radius. So we can say that $L(r) = L_{BH} = L_{Edd,es}$ everywhere.

(d) To find the escape time t_{esc} for the optically thick case, we know that photon dispersion $\propto N^2$, where $N \approx R/\lambda_{mfp}$. So

$$t_{esc} = (\text{dispersion}) \frac{\lambda_{mfp}}{c} = \frac{R^2}{{\lambda_{mfp}}^2} \frac{\lambda_{mfp}}{c} = \frac{R^2}{\lambda_{mfp} c} = \frac{R^2 \alpha}{c}$$

where $\alpha = n\sigma = \rho\kappa$. Since $\tau_0 = \rho_0 \kappa R_{out} = \alpha R_{out}$, we can write $\rho_0 = \tau_0/(\kappa R_{out})$, so $\alpha = \tau_0/R_{out}$. So t_{esc} becomes

$$t_{esc} = \frac{{R_{out}}^2}{c} \frac{\tau_0}{R_{out}}$$

$$\boxed{t_{esc} = \frac{R_{out}}{c} \tau_0}$$

At what value of τ_0 does our stationary approximation become questionable? For the stationary approximation to be valid, $t_{esc} \leq t_{dyn}$, and we know $t_{dyn} \approx 4.5 \times 10^{12}$ s $= 1.4 \times 10^5$ yr. We also know that $R_{out}/c \approx 10^9$ s, so $t_{esc} \approx (10^9 \text{ s})\, \tau_0$. So the stationary approximation breaks down if

$$\boxed{\tau_0 \geq 4.5 \times 10^3}$$

(e) Now we solve the diffusion equation for our optically thick case to determine $T(r)$.

$$L(r) = L_{BH} = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial}{\partial r} u(r)$$

Now $\kappa\rho = \alpha = \tau_0/R_{out} \rightarrow \rho = \tau_0/(R_{out}\kappa)$. Given the setup of our problem, we're assuming that ρ is constant. So

$$L_{BH} = -4\pi r^2 \frac{c R_{out}}{3\tau_0} \frac{\partial}{\partial r} u(r)$$

$$\frac{\partial}{\partial r} u(r) = -\frac{3\tau_0}{c R_{out}} \frac{L_{BH}}{4\pi r^2}$$

Integrate with respect to r:

$$u(r) = \frac{3\tau_0}{cR_{out}} \frac{L_{BH}}{4\pi r} + C$$

(where we will specify C later with our radiative zero boundary condition).

Now we know that $u(R) = 4\pi/cJ(r)$, so $J(r) = c/(4\pi)u(r)$, and since for grey absorptivity, we have

$$\begin{aligned} T_{equil}(r) &= \left[\frac{pi}{\sigma_{SB}} J(r)\right]^{1/4} \\ &= \left[\frac{\cancel{\pi}}{\sigma_{SB}} \frac{c}{4\cancel{\pi}} u(r)\right]^{1/4} \end{aligned}$$

$$T_{equil}(r) = \left[\frac{c}{4\sigma_{SB}} \left(\frac{3\tau_0}{cR_{out}} \frac{L_{BH}}{4\pi r} + C\right)\right]^{1/4}$$

Boundary condition: $T_{equil}(R_{out}) = 0$, so $C = -3\tau_0 L_{BH}/(4\pi c {R_{out}}^2)$. Now we get

$$T_{equil}(r) = \left[\frac{\cancel{c}}{4\sigma_{SB}} \left(\frac{3\tau_0}{\cancel{c}R_{out}} \frac{L_{BH}}{4\pi r} - \frac{3\tau_0}{\cancel{c}R_{out}} \frac{L_{BH}}{4\pi R_{out}}\right)\right]^{1/4}$$

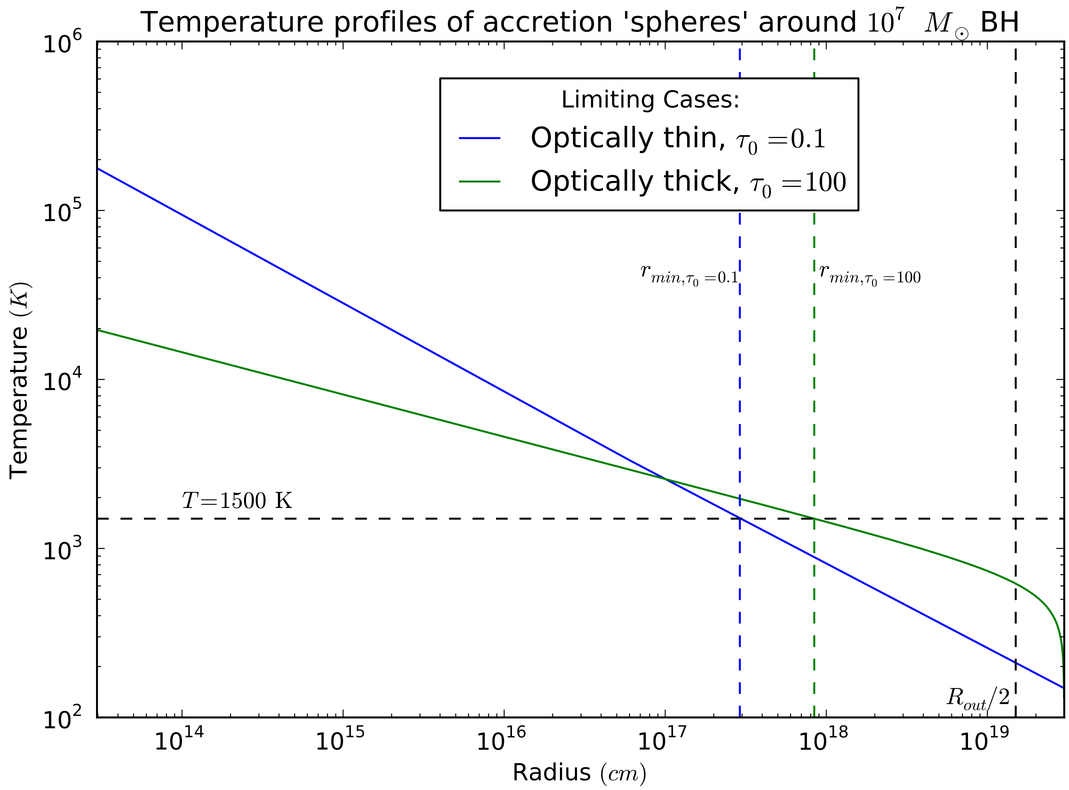
$$T_{equil}(r) = \left[\frac{3}{16\pi} \frac{\tau_0}{R_{out}} \frac{L_{BH}}{\sigma_{SB}} \left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}$$

But we also know that $L_{BH} = 4\pi {R_{in}}^2 \sigma_{SB} T_s^4$, so we can rewrite this expression as

$$T_{equil}(r) = \left[\frac{3}{4(\cancel{4\pi})} \frac{\tau_0}{R_{out}} \frac{\cancel{4\pi} {R_{in}}^2 \cancel{\sigma_{SB}} T_s^4}{\cancel{\sigma_{SB}}} \left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}$$

$$\boxed{T_{equil}(r) = T_s \left[\frac{3}{4} \frac{\tau_0}{R_{out}} {R_{in}}^2 \left(\frac{1}{r} - \frac{1}{R_{out}}\right)\right]^{1/4}}$$

(f) Plotting the temperature profiles for the optically thin and the optically thick cases give us the profiles below, using $\tau_0 = 0.1$ for the optically thin case and $\tau + 0 = 100$ for the optically thick case.



Because dust sublimates for $T \gtrsim 1500\,\mathrm{K}$, we find that this happens at about

$$\boxed{r_{min,\tau_0=0.1} \approx 2.9 \times 10^{17}\,\mathrm{cm} = 0.10\,\mathrm{pc}}$$

$$\boxed{r_{min,\tau_0=100} \approx 8.4 \times 10^{17}\,\mathrm{cm} = 0.28\,\mathrm{pc}}$$

(g) Thinking about our dusty gas in the middle of the envelope at $r = R_{out}/2$, we compare the ratio for the temperatures for the optically thick and the optically thin cases.

$$\frac{T_{opt\ thick}(R_{out}/2)}{T_{opt\ thin}(R_{out}/2)} = \frac{T_s\left[\frac{3}{4}\frac{\tau_0}{R_{out}}{R_{in}}^2\left(\frac{2}{R_{out}} - \frac{1}{R_{out}}\right)\right]^{1/4}}{T_s\left[\frac{1}{2}\left(1 - \left[1 - \frac{R_{in}^2}{(R_{out}/2)^2}\right]^{1/2}\right)\right]^{1/4}}$$

$$= \left[\frac{\frac{3}{4}\tau_0\frac{{R_{in}}^2}{{R_{out}}^2}}{\frac{1}{2}\left(1 - \left(1 - \frac{4{R_{in}}^2}{{R_{out}}^2}\right)^{1/2}\right)}\right]^{1/4}$$

Since $R_{in} \ll R_{out}/2$, we can write

$$\begin{aligned}\frac{1}{2}\left(1-\left(1-\frac{4{R_{in}}^2}{{R_{out}}^2}\right)^{1/2}\right) &\approx \frac{1}{2}\left[1-\left(1-\frac{1}{2}\frac{4{R_{in}}^2}{{R_{out}}^2}+\dots\right)\right] \\ &\approx \frac{1}{4}\frac{4{R_{in}}^2}{{R_{out}}^2} \\ &= \frac{{R_{in}}^2}{{R_{out}}^2}\end{aligned}$$

So our expression simplifies to

$$\frac{T_{opt\ thick}(R_{out}/2)}{T_{opt\ thin}(R_{out}/2)} \approx \left[\frac{\frac{3}{4}\tau_0\frac{{R_{in}}^2}{{R_{out}}^2}}{\frac{{R_{in}}^2}{{R_{out}}^2}}\right]^{1/4}$$

$$\boxed{\frac{T_{opt\ thick}(R_{out}/2)}{T_{opt\ thin}(R_{out}/2)} \approx \left[\frac{3}{4}\tau_0\right]^{1/4}}$$

For our two values, we have

$$\boxed{\begin{aligned}T_{opt\ thick}(R_{out}/2) &= 619\ \mathrm{K} \\ T_{opt\ thin}(R_{out}/2) &= 210\ \mathrm{K}\end{aligned}}$$

Now energy density in photons goes as $u = aT^4$, so with this ratio of temperatures we see that

$$\boxed{\frac{u_{opt\ thick}}{u_{opt\ thin}}(R_{out}/2) \approx \frac{3}{4}\tau_0}$$

It makes sense that the optically thick region would be hotter at R_{out}, since $L_{BH} = L(r)$ throughout the envelope, since optically thick implies that it traps energy. Also, you would expect that the more optically thick a region is, the more energy it traps, so the hotter the region is, and so you would expect u to scale as τ_0. This is because more opaque regions hold radiation in for a factor of τ_0 longer.

The emergent spectrum

(h) To find the emergent spectrum, we want to calculate the observed wavelength-specific luminosity.

First we start with the emissivity, j_ν (or rather we want j_λ, for the wavelength-dependent emissivity), where j_λ has units of erg s^{-1} cm^{-3} cm^{-1} str^{-1}. We will integrate over the solid angle and volume of the source to get the observed wavelength-specific luminosity.

Here, since we're assuming the envelope radiates isotropically, our integral over $\int d\Omega \to \times 4\pi$.

Case 1: Optically thin

$$\frac{j_\lambda}{\alpha} = B_\lambda, \qquad \alpha = \frac{\tau_0}{R_{out} - R_{in}} \approx \frac{\tau_0}{R_{out}}$$

$$B_\lambda(T) = B_\lambda(T(r)) = \frac{2hc}{\lambda^5}\frac{1}{e^{hc/(kT\lambda)}-1},$$

$$\text{So} \quad j_\lambda = \frac{\tau_0}{R_{out}}\frac{2hc}{\lambda^5}\frac{1}{e^{hc/(kT\lambda)}-1}$$

So we integrate to find

$$L_\lambda(\lambda) = \int_\Omega \int_{r_{min}}^{R_{out}} j_\lambda \, d\Omega \, dV = 4\pi \int_{r_{min} \approx 0.1\,\mathrm{pc}}^{R_{out}} j_\lambda 4\pi r^2 dr,$$

using $r_{min} \approx 0.1\,\mathrm{pc}$ because that is the radius outside of which the dust is not destroyed for the optically thin case.

For the numerical integration, take dr = constant spacing of $\approx 0.02\,\mathrm{pc}$ (actually choose to evaluate r at 500 points, so let it choose dr given the number of points over which to evaluate the integral.)

To carry out the numerial integration, first fix λ, then evaluate $L_\lambda(\lambda)$ at that λ. We will then loop over all λ from 1 to 200 μm, as this is an interesting IR range to consider, and then we plot $L_\lambda(\lambda)$ vs λ.

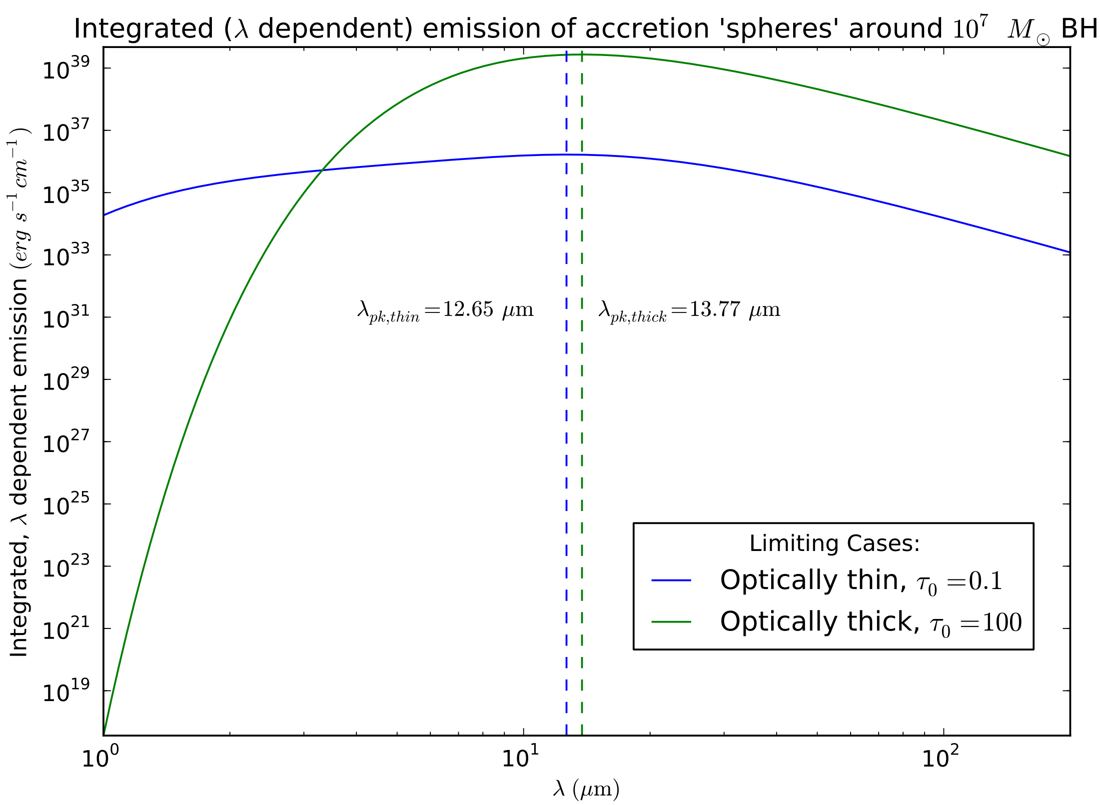
Case 2: Optically thick

$$L_\lambda(\lambda) = \frac{\tau_0}{R_{out}} B_\lambda(T_{eff}) \left(\frac{4\pi}{3} {R_{out}}^3 \right) (4\pi \text{ str}),$$

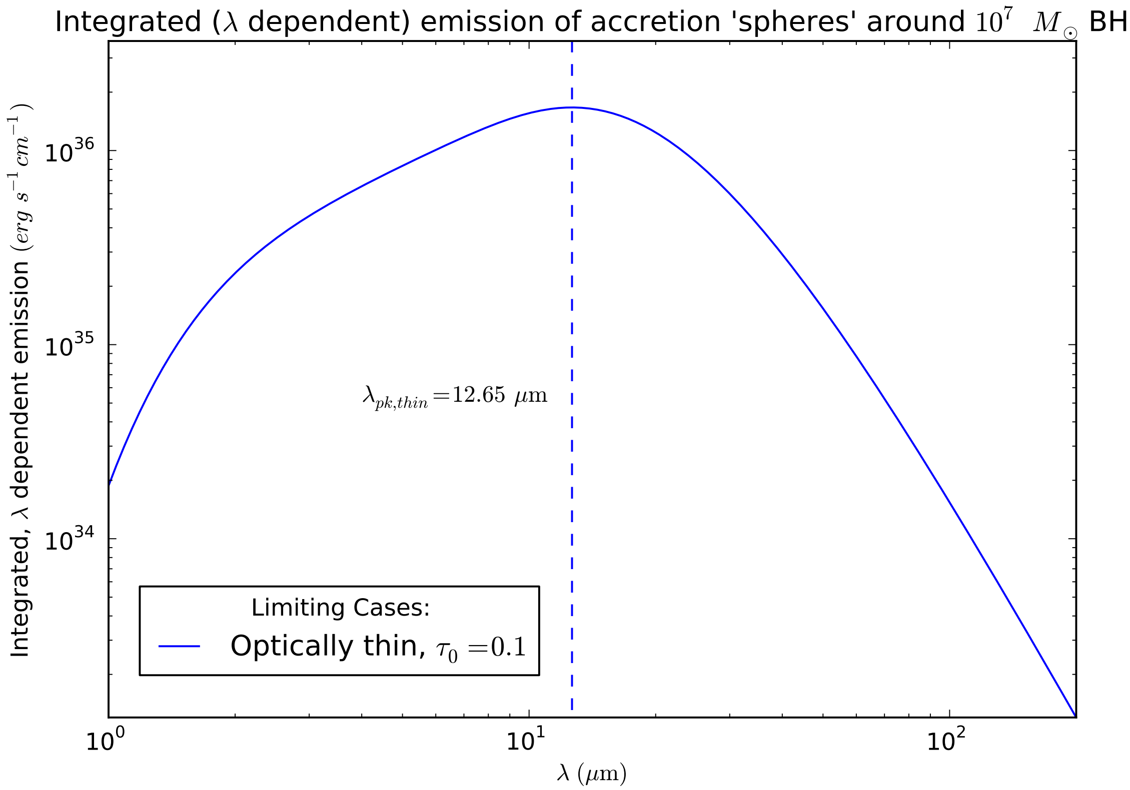
$$\text{where} \quad T_{eff} = \left(\frac{L_{BH}}{4\pi {R_{out}}^2 \sigma_{SB}} \right)^{1/4}$$

$$\boxed{T_{eff} = 210.4 \text{ K}}$$

After carrying out the numerical integration for the optically thin case, and the (relatively) simple calculation for the optically thick case, we get the emission profiles as plotted below.



To better show the structure of the optically thin emission curve, a zoom of the range of the optically thin emission curve is given below.



(The code S. P. used to do the numerical integration for the optically thin case is included as an appendix at the end of these solutions.)

Problem 2 - A Solar Supernova

a)

I use the Thomson cross-section to define τ because the remnant is mostly ionized hydrogen, and so optical depth is due mostly to electron scattering.

$$\tau = R_0 \alpha$$

$$\alpha \sim n\sigma_{Thomson}$$

$$n = \frac{\rho}{m_p} = \frac{\frac{M}{m_p}}{\frac{4}{3}\pi R^3}$$

$$\rightarrow \tau \simeq \frac{\frac{M}{m_p}\sigma_T}{\frac{4}{3}\pi R^3}$$

Plugging in $M \sim M_\odot$ and $R \sim R_\odot$, you get:

$$\tau \sim 10^{10}$$

b)

Let's assume the burst deposits one half the total energy into kinetic energy (bulk motion of the particles) and the other half into internal energy (thermal motion of the particles). First, we'll assume that the temperature is, to first order, set by the thermal radiation from the internally-deposited energy to find T.

$$U = aT^4$$

$$U \sim \frac{.5 \times 10^{51}\, ergs}{\frac{4}{3}\pi R^3}$$

$$T \sim \left(\frac{.5 \times 10^{51}\, ergs}{\frac{4}{3}\pi R^3 a}\right)^{\frac{1}{4}}$$

$$T \sim .8 \times 10^8 K$$

To check the assumption about which process sets the temperature (radiation or internal gas energy), let's check to make sure that the internal radiative energy is much larger than the internal gas energy at the temperature found above.

$$U_{radiative} \sim \frac{.5 \times 10^{51}\, ergs}{\frac{4}{3}\pi R^3} \sim 3.5 \times 10^{17}\, \frac{ergs}{cm^3}$$

$$U_{gas} \sim \frac{3}{2}nkT$$

$$n = \frac{\frac{M}{m_p}}{\frac{4}{3}\pi R^3}$$

$$U_{gas} \sim 1.4 \times 10^{16}\, \frac{ergs}{cm^3}$$

And so we see that the radiation energy density is more than an order of magnitude larger than the gas energy density, and we can safely assume that radiation energy dominates.

c)

In adiabatic expansion, no energy is lost from the system:

$$\frac{\partial E}{\partial t} = -P\frac{\partial V}{\partial t}$$

In radiation-dominated systems:

$$P(t) = \frac{U(t)}{3};\ E(t) = U(t)V(t)$$

Plug in to get:

$$\frac{\partial(U(t)V(t)}{\partial t} = \frac{U(t)}{3}\frac{\partial V(t)}{\partial t}$$

$$V(t)\frac{\partial U(t)}{\partial t} = \frac{-4}{3}U(t)\frac{\partial V(t)}{\partial t}$$

Assume a blackbody:

$$V(t)4aT^3(t)\frac{\partial T(t)}{\partial t} = \frac{-4}{3}aT^4(t)\frac{\partial V(t)}{\partial t)}$$

$$\frac{1}{T} = \frac{-1}{3V}\partial V$$

Integrate to get:

$$T = Ce^{-\frac{ln(V(t))}{3}}$$

$$V(t) = V_0\left(\frac{t}{t_0}\right)^3$$

$$T = CV_0^{-\frac{1}{3}}\left(\frac{t}{t_0}\right)^{-1}$$

Normalize to T_0 to get:

$$T(t) = T_0\left(\frac{t}{t_0}\right)^{-1}$$

To estimate the temperature of the debris, let's make some barely-justified assumptions. First, assume that the characteristic velocity of the debris is set by the kinetic energy:

$$v \sim \sqrt{2E/M} \sim 7 \times 10^8\,\frac{cm}{s}$$

An estimate for the time for the rain to get to earth is roughly d/v where d is the distance from the Sun and v is the velocity above. Using $d = 1.48 \times 10^{13}\,cm$ I get:

$$t \sim 2 \times 10^4\,s\ \sim 5.8\,hours$$

We know the temperature of the remnant from part (b) above, which is (roughly) the temperature when the shock wave hits the surface of the star.

$$t_0 \sim \frac{R_\odot}{v}\ \sim 98\,s$$

Plug it all in to find:

$$T(deadly\,rain) \sim 8.5 \times 10^6 K\left(\frac{2 \times 10^4\,s}{100\,s}\right)^{-1}$$

$$T \sim 4 \times 10^4\,K$$

d)

Time to do some math. Start with the diffusion approximation from problem 1.

$$\frac{u}{r} \simeq \frac{-u(t)}{R(t)}$$

$$L(t) = -4\pi r^2 \frac{c}{3\kappa\rho}\frac{\partial U(r)}{\partial r} \simeq -4\pi r^2 \frac{c}{3\kappa\rho}\frac{-u(t)}{R(t)}$$

Plug in $R(t) \sim r(t) \simeq vt$, and $\rho(t) \simeq M/V(t)$:

$$L(t) = \frac{4\pi vtc}{3\kappa M} V(t)U(t)$$

Now that we have a form for $L(t)$ let's go back to the energy equation:

$$\frac{\partial E}{\partial t} = -P\frac{\partial V}{t} - L(t)$$

Use $E = U \times V$, and note that I'm switching to the dot-derivative notation for readability.

$$\dot{U}V + U\dot{V} = \frac{-U\dot{V}}{3} - L$$

$$\dot{U}V = \frac{-4}{3}U\dot{V} - L$$

$$V = V_0\left(\frac{t}{t_0}\right)^3 \rightarrow \dot{V} = \frac{3V}{t}$$

$$\dot{U} = \frac{-4U}{t} - \frac{L}{V}$$

Plug in $L(t)$ from above (note that a factor of $V(t)$ cancels out):

$$\frac{\partial U}{\partial t} = \frac{-4U}{t} - \frac{4\pi vtc}{3\kappa M}U$$

$$\frac{\partial U}{U} = \frac{-4\partial t}{t} - \frac{4\pi vc}{3\kappa M}t\partial t$$

Integrate to get:

$$ln\left(\frac{U}{U_0}\right) = -4ln\left(\frac{t}{t_0}\right) - \frac{2\pi vc}{3\kappa M}(t^2 - t_0^2)$$

$$U(t) = U_0\left(\frac{t}{t_0}\right)^4 e^{-\frac{2\pi vc}{3\kappa M}(t^2 - t_0^2)}$$

Plug that back into our equation for luminosity, using $U_0 = E/V_0$ and $V(t) = V_0(t/t_0)^3$, and notice that all of the time terms cancel except for those in the exponential:

$$L(t) = \frac{4\pi vc}{3\kappa M}Et_0 e^{-\frac{2\pi vc}{3\kappa M}(t^2 - t_0^2)}$$

Note that the luminosity scale is really set by a constant out front, plugging in $R_0 = vt_0$:

$$L_{sn} \sim \frac{4\pi cER_0}{3\kappa M}$$

To find the characteristic time scale, I find the time at which the time-varying exponent is equal to -1. Note that I implicitly pull the constant part of the exponential out.

$$\tau_{sn} \sim \sqrt{\frac{3\kappa M}{2\pi vc}}$$

e)

boom
Using electron-scattering and a $\kappa \sim 0.4$, I calculate our supernova to have a luminosity of about 10^{40} ergs per second, or $10^6 L_\odot$. It has a lifetime of about 4×10^6 seconds, or 42 days. See the plot below!

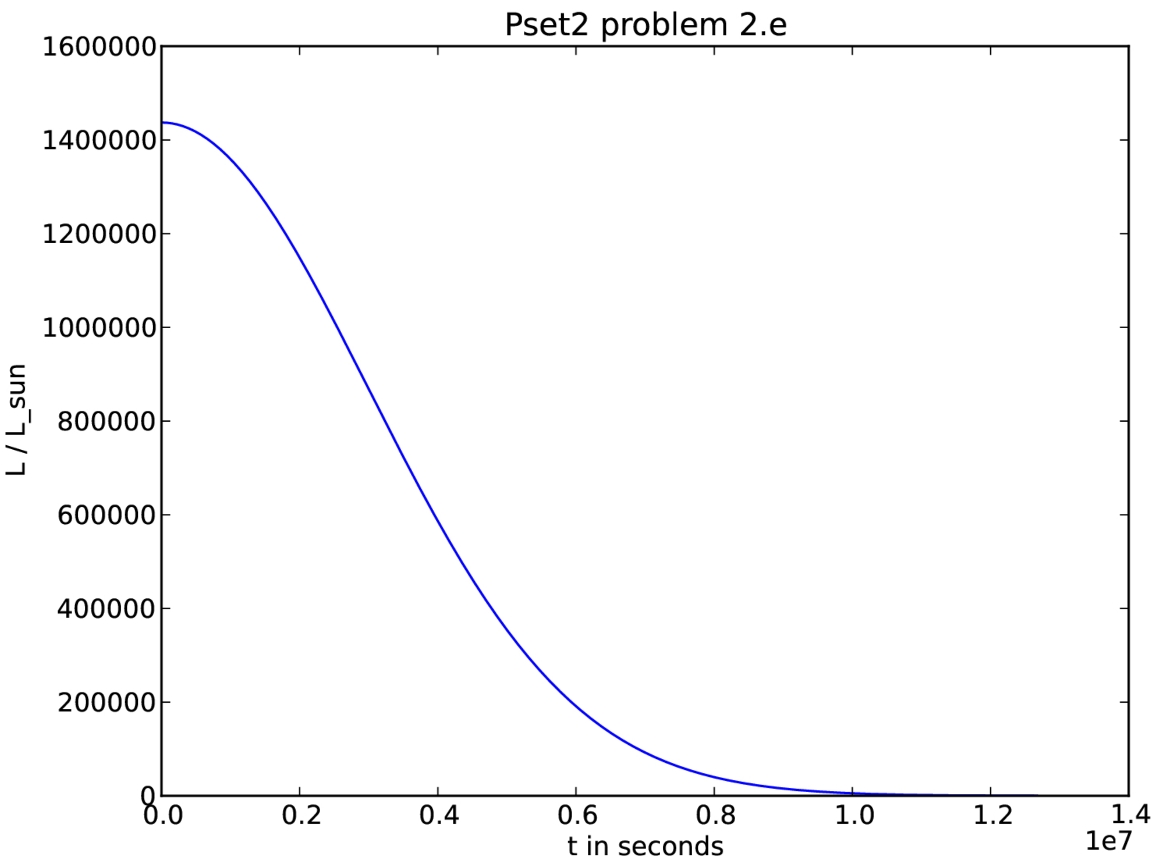


Figure 1: The evolution of our supernova's lightcurve.

g (the rebel one - he doesn't care if he's out of order)

$$E_{sn} \sim L_{sn}\tau_{sn} \simeq 4 \times 10^{45}\, ergs$$

This is a tiny fraction of the original burst energy. Most of that initial energy must go into bulk motion of the gas (kinetic energy) or into the binding energy of large atoms, some of which we later see as the radioactive decay of nickel.

f

Assume a blackbody:

$$L_{sn} = 4\pi R_0^2 \sigma_B T^4$$

$$T_{sn} \simeq \left(\frac{L_{sn}}{4\pi\sigma_B R_0^2}\right)^{\frac{1}{4}} \simeq 8.7 \times 10^4\, K$$

$$\lambda \simeq \frac{.29\, cm\, K^{-1}}{T} \simeq 3.3 \times 10^{-6}\, cm$$

Our supernova peaks in the optical or UV, at about 300 nm.

g (again)

The luminosities of about 10^{42} ergs per second are two orders of magnitude brighter than the model explained above. Noting that R is actually about $100R_\odot$ goes a long way towards explaining this discrepancy, since $L \sim R$.

Appendix to Prob. 1: Code used for numerical integration and plotting of the emission curves

This code was written in python, and run using ipython with pylab.

```
import numpy as np
import scipy as sp
import matplotlib.pyplot as plt

autocm = 1.5e13         # conversion from  AU to cm
pctocm = 3.e18          # conversion from  pc to cm
kpctocm = 3.e21         # conversion from kpc to cm
Mpctocm = 3.e24         # conversion from Mpc to cm
cmtoaa = 1.e8           # conversion from  cm to AA
yr = 3.15e7             # conversion from  yr to sec

Msun = 2.e33            # g

#constants
me = 9.1e-27            # g                       ## mass electron
sigmaT = 6.65e-25       # cm^2                    ## Thomson xsection
sbconst = 5.67e-5       # erg/cm^2/s/K^2          ## Stefan-Boltzmann constant
c = 3.e10               # cm/s
G = 6.67e-8             # cm^3/s^2/g
kappaes = .4            # cm^2/g                  ## opacity for electron scattering
kb = 1.38e-16           # erg/K                   ## Boltzmann constant
h = 6.626e-27           # erg s                   ## Planck constant


def Toptthin(r,Ts,Rin):
  return Ts*pow(.5*(1-sqrt(1-Rin**2/r**2)),1./4.)

Mbh = 1.e7*Msun
Rin = sqrt(400*G**2*Mbh**2/pow(c,4.))
Rout = 10*pctocm        # 10 pc -> cm

Ts = pow(pow(c,5.)/(400.*sbconst*G*Mbh*kappaes),1./4.)

## Optically thick case

tau0max = tff/(Rout/c)

def Toptthick(r,Ts,Rin,Rout,Tau0):
  return Ts*pow(3./4.*Tau0/Rout*Rin**2*(1./r-1./Rout),1./4.)


rarr = np.linspace(Rin,Rout,500)
Tthickarr = Toptthick(rarr,Ts,Rin,Rout,100.)
Tthinarr = Toptthin(rarr,Ts,Rin)
dustlimit = rarr*0.+1500.
rhalf = rarr*0.+Rout/2.
temprange= np.linspace(100.,1.e6,500)

# Where does dust sublimation happen?
minthin = max(rarr.compress((Tthinarr>1500.).flat))     # max for opt thin  disk
minthick = max(rarr.compress((Tthickarr>1500.).flat))   # max for opt thick disk
```

```
minthin = minthin+5.e16         # actually closer to the T = 1500 K cutoff


# Numerical integration for part III of #1.
lamarrthin = np.logspace(-4,-4+log10(200.),500)
tau0thin = 0.1

specthinarr = []
rarr = np.linspace(minthin,Rout,500)
dr = rarr[1]-rarr[0]

def Bplanck(lam,Teff):
  return 2*h*c/pow(lam,5.)*1/(exp(h*c/(kb*Teff*lam)) - 1)

def jlamr(lam,r):
  return tau0thin/(Rout-minthin)*2*h*c/lam**5*1./(exp(h*c/(kb*Toptthin(r,Ts,Rin)*lam))-1)

# Integrate
for lam in lamarrthin:
  tmp = 0.
  for r in rarr:
    tmp = tmp + 4*pi*jlamr(lam,r)*(4*pi*r**2*dr)
  specthinarr.append(tmp)

# Change it to a np array
specthinarr = np.array(specthinarr)


# Optically thick case: BB with Teff
Teff = pow(Lbh/(4*pi*Rout**2*sbconst),.25)

# default to 1 to 200 um?
#lamarrthick = np.linspace(lambdapk(Toptthin(minthin,Ts,Rin)),lambdapk(Toptthin(Rout,Ts,Rin)),500)
lamarrthick = np.logspace(-4,-4+log10(200.),500)
#lamarrthick = np.linspace(1.e-4,200.e-4,500)
tau0thick = 100.

specthickarr = tau0thick/(Rout-minthick)*Bplanck(lamarrthick,Teff)*4*pi/3.*
(Rout**3-minthick**3)*4*pi    # "integrate" over volume, solid angle


## Plotting
fig = plt.figure()
ax = fig.add_subplot(111)

ax.set_title("Integrated ($\lambda$ dependent) emission of accretion 'spheres' around
$10^7\, M_{\odot}$ BH")
ax.set_xlabel('$\lambda \ (\mathrm{\mu m})$')
ax.set_ylabel('Integrated, $\lambda$ dependent emission $(erg \, s^{-1} cm^{-1})$')
ax.set_xscale('log')
ax.set_yscale('log')
```

```
pkthinwhere = where(specthinarr==max(specthinarr))
pkthinind = pkthinwhere[0][0]
pkthin = lamarrthin[pkthinind] + lamarrthin*0.

pkthickwhere = where(specthickarr==max(specthickarr))
pkthickind = pkthickwhere[0][0]
pkthick = lamarrthick[pkthickind] + lamarrthick*0.

temprange = np.linspace(min(specthickarr),max(specthickarr)+2.e39,500)

l1,l2,thinpk,thickpk = ax.plot(lamarrthin*1.e4,specthinarr, 'b-',
lamarrthick*1.e4, specthickarr, 'g-',pkthin*1.e4,temprange,'b--',
pkthick*1.e4,temprange,'g--')

fig.legend((l1,l2),('Optically thin, $\\tau_0 = 0.1$', 'Optically thick, $\\tau_0 = 100$'),
(.5,.2),title='Limiting Cases:')


text(4,1.e31,'$\lambda_{pk,thin} = 12.65 \, \mathrm{\mu m}$')
text(15,1.e31,'$\lambda_{pk,thick} = 13.77 \, \mathrm{\mu m}$')

ax.set_xbound([min(lamarrthin*1.e4),max(lamarrthin*1.e4)])
ax.set_ybound([min(specthickarr),max(specthickarr)+2.e39])

plt.show()
```